30/11/23

MATH4030 Tutorial

Announcements:

- Aim to post HW5 Sol'ns today
- Final: 20/12.

Q3 of HW5:

1^{st} ff. of cone using F: $\begin{bmatrix} 1 & 0 \\ 0 & u^2\sin^2\alpha \end{bmatrix}$.

U, using identity param. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but can param. U using polar coordinates

$X(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, 0)$, $\rho = u\sin\alpha > 0$, $0 < \theta < 2\pi\sin\alpha$.

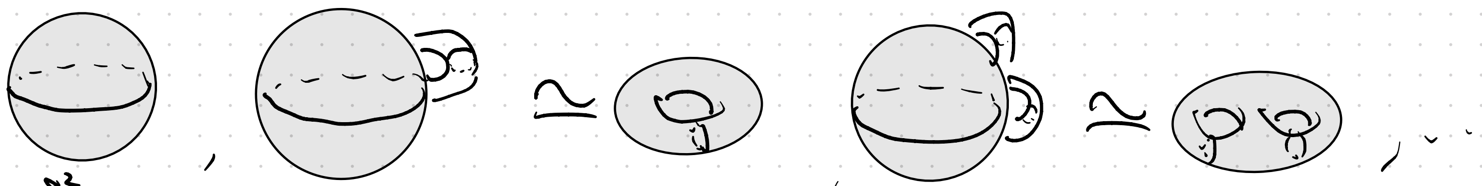
Then 1^{st} ff. $\begin{bmatrix} 1 & 0 \\ 0 & \rho^2 \end{bmatrix}$

if $\exists$ param. $X: U \to S$, $\bar{X}: U \to \bar{S}$ s.t. 1^{st} ff. are the same, then $\varphi = \bar{X} \circ X^{-1}$ is a local isometry

This makes the map $F \circ X^{-1}$ a local isometry. (Prop. 1 of 4-2 do Carmo)

Facts from Topology:

1) Classification of all compact connected surfaces $S^2 \subseteq \mathbb{R}^3$: Let S be a compact connected surface in $\mathbb{R}^3$. Then S is homeomorphic to a sphere with a number of handles g (genus) attached and $\chi(S) = 2-2g$, $g=0,1,2,\dots$



S^2, $\chi(S)=0$, $\chi(S)=-2$

$g=0$.

$\chi(S^2)=2$

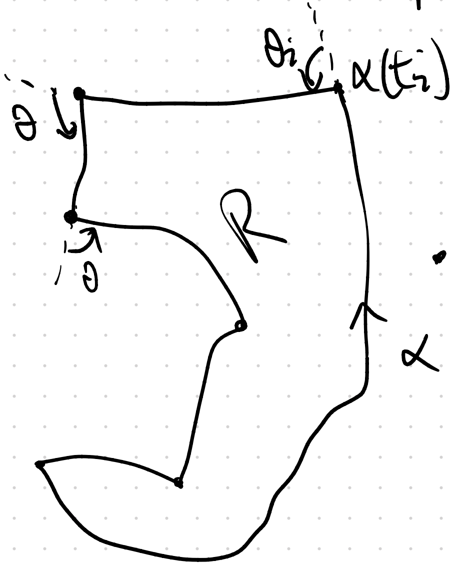
Def: $\alpha:[0,l] \to S$ a cts path is

1) Closed: $\alpha(0)=\alpha(l)$

2) Simple: if for $t_1 \neq t_2$, $t_1, t_2 \in [0,l)$ then $\alpha(t_1) \neq \alpha(t_2)$.

α has no self-intersections

3) Piecewise regular: $\exists$ subdivision $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = l$ s.t. α is regular and smooth in each (t_i, t_{i+1}) $i = 1, \ldots, k$.



$\alpha(t_i)$ are called vertices, θ_i is the external angle at vertex $\alpha(t_i)$.

- $R \subseteq S$ is a simple region if R is homeomorphic to the disk ($\chi(R) = 1$), and ∂R is the trace of a simple, closed, piecewise regular curve.

- $R \subseteq S$ is regular if R is compact and ∂R is a finite union of simple, closed, piecewise regular curves which do not intersect.

Jordan Curve Thm: Every closed piecewise regular curve in the plane is the boundary of a simple region.

Global GB: let $R \subset S$ be a regular region of an oriented surface and let $C_1, \dots, C_n$ be the simple closed piecewise regular curves which form ∂R, $\theta_1, \dots, \theta_p$ be the external angles. Then

$$\sum_{i=1}^{k} \int_{C_i} k_g(s)\,ds + \iint_R K\,dA + \sum_{j=1}^{p} \theta_j = 2\pi \chi(R).$$

- If R is simple, then $\chi(R)=1$, so $\displaystyle\sum_{i=1}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\,ds + \iint_R K\,dA + \sum_{i=1}^{k} \theta_i = 2\pi$

Note: Def: a surface S is closed if it is compact and has no boundary.

- If S is an orientable, closed surface, then it can be viewed as a region w/out boundary, then $\displaystyle\iint_S K\,dA = 2\pi \chi(S)$.

Q1: Any closed surface with everywhere positive curvature is homeomorphic to the sphere.

Pf: $K > 0$ everywhere, so $0 < \iint_S K\,dA = 2\pi\chi(S)$. $\Rightarrow \chi(S) > 0$.

and by classification Thm, the sphere is the only compact surface w/ $\chi(S) > 0$. So $S \simeq$ Sphere.

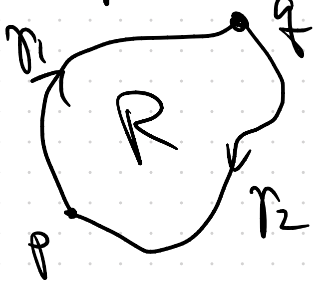
Q2: Let S be a closed surface not homeomorphic to the sphere. Then K attains both positive and negative values.

Pf: Since S is closed, $\exists\, p_0 \in S$ s.t. $K(p_0) > 0$.

OTOH, $\chi(S) \leq 0 \Rightarrow \iint_S K\,dA \leq 0$, so K must also attain negative values.

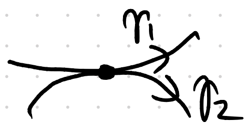
Q3: Let S be an orientable surface w/ everywhere $K \leq 0$. Then two geodesics γ_1, γ_2 which start from $p \in S$ cannot meet again at $q \in S$, $q \neq p$ s.t. $\gamma_1 \cup \gamma_2$ form the boundary of a simple region $R \subseteq S$.

Pf: Suppose the contrary.



$$\sum_{i=1}^{2} \int_{\gamma_1,\gamma_2} \overset{0}{k_g(s)}\,ds + \iint_R K\,dA + \theta_1 + \theta_2 = 2\pi.$$

By simple,



cannot occur, $\theta_i < \pi$ for $i = 1, 2$.

So $\iint_R K\,dA = 2\pi - \theta_1 - \theta_2 > 0$, which contradicts $K \leq 0$.

On a surface w/ $K \leq 0$, $\nexists$ simple closed geodesics which is the boundary of a simple region.

Q4 If S is a closed surface w/ $K>0$ and γ_1, γ_2 are simple closed geodesics on S, then γ_1, γ_2 intersect everywhere

Pf: By Q1, $S \cong$ Sphere. Sps γ_1, γ_2 do not intersect.

Then $\gamma_1 \cup \gamma_2$ is the boundary of a region R w/ $\chi(R) = 0$

So $0 = 2\pi\chi(R) = \iint_R K\,dA$, but this contradicts $K>0$ everywhere. /.